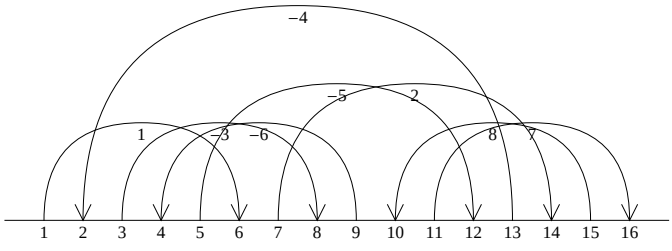
Pensieve Header: A sub-meta-group of the $\beta$-meta-group?

```
<< KnotTheory`
GD[K_] := GD @@ (
    PD[K] /. X[i_, j_, k_, l_] :> If[PositiveQ[X[i, j, k, l]],
        Ar[l, i, +1], Ar[j, i, -1]
      ]
  )
```

```
Loading KnotTheory` version of August 22, 2010, 13:36:57.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Draw[expr_] := expr /. gd_GD :> Draw[gd];
Draw[gd_GD] := Module[
   {n = Length[gd], h, k = 0},
   Graphics[{
     Line[{{0, 0}, {2 n + 1, 0}}],
     Table[Text[i, {i, -0.3}], {i, 2 n}],
     (List @@ gd) /. {
       Ar[i_, j_, s_] :> {
         h = Abs[i - j] / 2;
         BezierCurve[{
           {i, 0}, {i, h}, {(i + j) / 2, h}, {j, h}, {j, 0}
          }, SplineDegree → 2],
         Text[s * (++k), {(i + j) / 2, h - 0.3}],
         Line[{{j - 0.2, 0.4}, {j, 0}, {j + 0.2, 0.4}}]
        }
      }
    }
   ]
  ];
Draw[GD[Knot[8, 17]]]
```



```
βSimplify = Factor;
SetAttributes[βCollect, Listable];
βCollect[B[ω_, μ_]] := B[
   βSimplify[ω],
   Collect[μ, _h, Collect[#, _t, βSimplify] &]
  ];
(* "L" for "Labels" *)
hL[β_] := Union[Cases[β, h[s_] :> s, Infinity]];
tL[β_] := Union[Cases[β, t[s_] | c_s_ :> s, Infinity]];
dL[β_] := Union[hL[β], tL[β]];
SetAttributes[βForm, Listable];
βForm[B[ω_, μ_]] := Module[
   {tails, heads, mat},
   tails = tL[B[ω, μ]]; heads = hL[B[ω, μ]];
   mat = Outer[βSimplify[Coefficient[μ, h[#1] t[#2]]] &, heads, tails];
   PrependTo[mat, t /@ tails];
   mat = Prepend[Transpose[mat], Prepend[h /@ heads, ω]];
   MatrixForm[mat]
  ];
R[x_, y_] := B[1, (E^c_x - 1) / c_x * t[x] h[y]];
Rinv[x_, y_] := B[1, (E^(-c_x) - 1) / c_x * t[x] h[y]];
tm[x_, y_, z_][β_] := β /. {t[x] → t[z], t[y] → t[z], c_x → c_z, c_y → c_z};
hm[x_, y_, z_][B[ω_, μ_]] := Module[
   {γx = D[μ, h[x]], γy = D[μ, h[y]], M = μ /. h[x] | h[y] → 0},
   B[ω, M + h[z] (γx + γy + (γx /. t[i_] :> c_i) γy)] // βCollect
  ];
swap[x_, y_][B[ω_, μ_]] := Module[
   {α, β, γ, δ, ϵ},
   α = Coefficient[μ, h[x] t[y]];
   β = D[μ, t[y]] /. h[x] → 0;
   γ = D[μ, h[x]] /. t[y] → 0;
   δ = μ /. h[x] | t[y] → 0;
   ϵ = 1 + c_y α;
   B[ω * ϵ, Plus[
      α (1 + (γ /. t[i_] :> c_i) / ϵ) h[x] t[y],
      β (1 + (γ /. t[i_] :> c_i) / ϵ) t[y],
      γ / ϵ h[x],
      δ - c_y / ϵ γ * β
     ]] // βCollect
  ];
gm[x_, y_, z_][β_] := β // swap[y, x] // hm[x, y, z] // tm[x, y, z];
B /: B[ω1_, μ1_] B[ω2_, μ2_] := B[ω1 * ω2, μ1 + μ2];
```

```
{
  β = B[ω[c1, c2, c3, c4], Sum[αi[c1, c2, c3, c4] t[i] h[1], {i, 4}]],
  β // tm[1, 2, 1],
  t1 = β // tm[1, 2, 1] // tm[1, 3, 1],
  t2 = β // tm[2, 3, 28] // tm[1, 28, 1],
  t1 == t2
 } // βForm
```

$$\left\{\begin{pmatrix} \omega[c_1, c_2, c_3, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_2, c_3, c_4] \\ t[2] & \alpha_2[c_1, c_2, c_3, c_4] \\ t[3] & \alpha_3[c_1, c_2, c_3, c_4] \\ t[4] & \alpha_4[c_1, c_2, c_3, c_4] \end{pmatrix},\right.$$

$$\begin{pmatrix} \omega[c_1, c_1, c_3, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_1, c_3, c_4] + \alpha_2[c_1, c_1, c_3, c_4] \\ t[3] & \alpha_3[c_1, c_1, c_3, c_4] \\ t[4] & \alpha_4[c_1, c_1, c_3, c_4] \end{pmatrix},$$

$$\begin{pmatrix} \omega[c_1, c_1, c_1, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_1, c_1, c_4] + \alpha_2[c_1, c_1, c_1, c_4] + \alpha_3[c_1, c_1, c_1, c_4] \\ t[4] & \alpha_4[c_1, c_1, c_1, c_4] \end{pmatrix},$$

$$\left.\begin{pmatrix} \omega[c_1, c_1, c_1, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_1, c_1, c_4] + \alpha_2[c_1, c_1, c_1, c_4] + \alpha_3[c_1, c_1, c_1, c_4] \\ t[4] & \alpha_4[c_1, c_1, c_1, c_4] \end{pmatrix}, \beta\text{Form}[\text{True}]\right\}$$

```
{
   β = B[ω, Sum[α10 i+j t[i] h[j], {i, 2}, {j, 4}]],
   β // hm[1, 2, 1],
   t1 = β // hm[1, 2, 1] // hm[1, 3, 1],
   t2 = β // hm[2, 3, 28] // hm[1, 28, 1],
   t1 == t2
  } // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] & h[4] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[3] & h[4] \\ t[1] & \alpha_{11} + \alpha_{12} + c_1 \alpha_{11} \alpha_{12} + c_2 \alpha_{12} \alpha_{21} & \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} + \alpha_{22} + c_1 \alpha_{11} \alpha_{22} + c_2 \alpha_{21} \alpha_{22} & \alpha_{23} & \alpha_{24} \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] \\ t[1] & \alpha_{11} + \alpha_{12} + c_1 \alpha_{11} \alpha_{12} + \alpha_{13} + c_1 \alpha_{11} \alpha_{13} + c_1 \alpha_{12} \alpha_{13} + c_1^2 \alpha_{11} \alpha_{12} \alpha_{13} + c_2 \alpha_{12} \alpha_{21} + c_2 \alpha_{13} \alpha_{21} + c_1 c_2 \alpha_{12} \\ t[2] & \alpha_{21} + \alpha_{22} + c_1 \alpha_{11} \alpha_{22} + c_2 \alpha_{21} \alpha_{22} + \alpha_{23} + c_1 \alpha_{11} \alpha_{23} + c_1 \alpha_{12} \alpha_{23} + c_1^2 \alpha_{11} \alpha_{12} \alpha_{23} + c_2 \alpha_{21} \alpha_{23} + c_1 c_2 \alpha_{12} \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] \\ t[1] & \alpha_{11} + \alpha_{12} + c_1 \alpha_{11} \alpha_{12} + \alpha_{13} + c_1 \alpha_{11} \alpha_{13} + c_1 \alpha_{12} \alpha_{13} + c_1^2 \alpha_{11} \alpha_{12} \alpha_{13} + c_2 \alpha_{12} \alpha_{21} + c_2 \alpha_{13} \alpha_{21} + c_1 c_2 \alpha_{12} \\ t[2] & \alpha_{21} + \alpha_{22} + c_1 \alpha_{11} \alpha_{22} + c_2 \alpha_{21} \alpha_{22} + \alpha_{23} + c_1 \alpha_{11} \alpha_{23} + c_1 \alpha_{12} \alpha_{23} + c_1^2 \alpha_{11} \alpha_{12} \alpha_{23} + c_2 \alpha_{21} \alpha_{23} + c_1 c_2 \alpha_{12} \end{pmatrix}$$

$\beta\text{Form}[\text{True}]$

```
Clear[β];
{β1 = B[ω, h[1] t[1] α + h[2] t[1] β + h[1] t[2] γ + h[2] t[2] δ],
  β1 // swap[1, 1]
 } // βForm
```

$$\left\{\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha & \beta \\ t[2] & \gamma & \delta \end{pmatrix}, \begin{pmatrix} \omega\,(1+\alpha\,c_1) & h[1] & h[2] \\ t[1] & \frac{\alpha\,(1+\alpha\,c_1+\gamma\,c_2)}{1+\alpha\,c_1} & \frac{\beta\,(1+\alpha\,c_1+\gamma\,c_2)}{1+\alpha\,c_1} \\ t[2] & \frac{\gamma}{1+\alpha\,c_1} & \frac{\delta-\beta\,\gamma\,c_1+\alpha\,\delta\,c_1}{1+\alpha\,c_1} \end{pmatrix}\right\}$$

```
{
   β = B[ω, Sum[α10 i+j t[i] h[j], {i, 2}, {j, 3}]],
   β // hm[1, 2, 1],
   t1 = β // hm[1, 2, 1] // swap[1, 1],
   t2 = β // swap[1, 1] // swap[2, 1] // hm[1, 2, 1],
   First[t1] == First[t2],
   Last[t1] == Last[t2] // Simplify
  } // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[3] \\ t[1] & \alpha_{11}+\alpha_{12}+c_1\,\alpha_{11}\,\alpha_{12}+c_2\,\alpha_{12}\,\alpha_{21} & \alpha_{13} \\ t[2] & \alpha_{21}+\alpha_{22}+c_1\,\alpha_{11}\,\alpha_{22}+c_2\,\alpha_{21}\,\alpha_{22} & \alpha_{23} \end{pmatrix}$$

$$\begin{pmatrix} \omega\,\left(1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}\right) & h[1] \\ t[1] & \frac{(1+c_1\,\alpha_{11}+c_2\,\alpha_{21})\,(\alpha_{11}+\alpha_{12}+c_1\,\alpha_{11}\,\alpha_{12}+c_2\,\alpha_{12}\,\alpha_{21})\,(1+c_1\,\alpha_{12}+c_2\,\alpha_{22})}{1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}} \\ t[2] & \frac{\alpha_{21}+\alpha_{22}+c_1\,\alpha_{11}\,\alpha_{22}+c_2\,\alpha_{21}\,\alpha_{22}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}} \end{pmatrix} \quad \frac{-c_1\,\alpha_1}{}$$

$$\begin{pmatrix} \omega\,\left(1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}\right) & h[1] \\ t[1] & \frac{(1+c_1\,\alpha_{11}+c_2\,\alpha_{21})\,(\alpha_{11}+\alpha_{12}+c_1\,\alpha_{11}\,\alpha_{12}+c_2\,\alpha_{12}\,\alpha_{21})\,(1+c_1\,\alpha_{12}+c_2\,\alpha_{22})}{1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}} \\ t[2] & \frac{\alpha_{21}+\alpha_{22}+c_1\,\alpha_{11}\,\alpha_{22}+c_2\,\alpha_{21}\,\alpha_{22}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}} \end{pmatrix} \quad \frac{-c_1\,\alpha_1}{}$$

```
βForm[True]
βForm[True]
```

```
{
   β = B[ω, Sum[α10 i+j t[i] h[j], {i, 3}, {j, 2}]],
   t1 = β // tm[1, 2, 1] // swap[1, 1],
   t2 = β // swap[1, 2] // swap[1, 1] // tm[1, 2, 1],
   First[t1] == First[t2],
   Last[t1] == Last[t2] // Simplify
  } // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11} & \alpha_{12} \\ t[2] & \alpha_{21} & \alpha_{22} \\ t[3] & \alpha_{31} & \alpha_{32} \end{pmatrix}$$

$$\begin{pmatrix} \omega\,(1 + c_1\,\alpha_{11} + c_1\,\alpha_{21}) & h[1] & h[2] \\ t[1] & \frac{(\alpha_{11}+\alpha_{21})\,(1+c_1\,\alpha_{11}+c_1\,\alpha_{21}+c_3\,\alpha_{31})}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} & \frac{(\alpha_{12}+\alpha_{22})\,(1+c_1\,\alpha_{11}+c_1\,\alpha_{21}+c_3\,\alpha_{31})}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} \\ t[3] & \frac{\alpha_{31}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} & \frac{-c_1\,\alpha_{12}\,\alpha_{31}-c_1\,\alpha_{22}\,\alpha_{31}+\alpha_{32}+c_1\,\alpha_{11}\,\alpha_{32}+c_1\,\alpha_{21}\,\alpha_{32}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} \end{pmatrix}$$

$$\begin{pmatrix} \omega\,(1 + c_1\,\alpha_{11} + c_1\,\alpha_{21}) & h[1] & h[2] \\ t[1] & \frac{(\alpha_{11}+\alpha_{21})\,(1+c_1\,\alpha_{11}+c_1\,\alpha_{21}+c_3\,\alpha_{31})}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} & \frac{(\alpha_{12}+\alpha_{22})\,(1+c_1\,\alpha_{11}+c_1\,\alpha_{21}+c_3\,\alpha_{31})}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} \\ t[3] & \frac{\alpha_{31}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} & \frac{-c_1\,\alpha_{12}\,\alpha_{31}-c_1\,\alpha_{22}\,\alpha_{31}+\alpha_{32}+c_1\,\alpha_{11}\,\alpha_{32}+c_1\,\alpha_{21}\,\alpha_{32}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} \end{pmatrix}$$

βForm[True]
βForm[True]

```
{
   β = B[ω, Sum[α10 i+j t[i] h[j], {i, 4}, {j, 4}]],
   t1 = β // gm[1, 2, 1] // gm[1, 3, 1],
   t2 = β // gm[2, 3, 2] // gm[1, 2, 1],
   First[t1] == First[t2],
   Last[t1] == Last[t2] // Simplify
  } // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] & h[4] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ t[4] & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix}$$

$$\begin{pmatrix} \omega\left(1 + c_1\,\alpha_{12} + c_1\,\alpha_{13} + c_1^2\,\alpha_{12}\,\alpha_{13} + c_1\,\alpha_{23} + c_1^2\,\alpha_{12}\,\alpha_{23} + c_1^2\,\alpha_{13}\,\alpha_{32} + c_1\,c_4\,\alpha_{13}\,\alpha_{42}\right) & \frac{\alpha_{11}+\alpha_{12}+2\,c_1\,\alpha_{11}\,\alpha_{12}+c_1\,\alpha_{12}^2+c_1^2\,\alpha_{11}}{} \\ t[1] & \\ t[4] & \end{pmatrix}$$

$$\begin{pmatrix} \omega\left(1 + c_1\,\alpha_{12} + c_1\,\alpha_{13} + c_1^2\,\alpha_{12}\,\alpha_{13} + c_1\,\alpha_{23} + c_1^2\,\alpha_{12}\,\alpha_{23} + c_1^2\,\alpha_{13}\,\alpha_{32} + c_1\,c_4\,\alpha_{13}\,\alpha_{42}\right) & \frac{\alpha_{11}+\alpha_{12}+2\,c_1\,\alpha_{11}\,\alpha_{12}+c_1\,\alpha_{12}^2+c_1^2\,\alpha_{11}}{} \\ t[1] & \\ t[4] & \end{pmatrix}$$

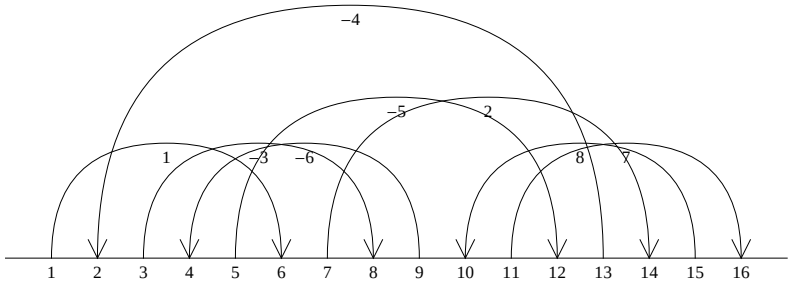
βForm[True]
βForm[True]

```
(β = Times @@ GD[K] /. {Ar[x_, y_, +1] :> R[x, y], Ar[x_, y_, -1] :> Rinv[x, y]}) //
 βForm
```

| 1 | h[2] | h[4] | h[6] | h[8] | h[10] | h[12] | h[14] | h[16] |
|---|---|---|---|---|---|---|---|---|
| t[1] | 0 | 0 | $\frac{-1+e^{c_1}}{c_1}$ | 0 | 0 | 0 | 0 | 0 |
| t[3] | 0 | 0 | 0 | $-\frac{e^{-c_3}\,(-1+e^{c_3})}{c_3}$ | 0 | 0 | 0 | 0 |
| t[5] | 0 | 0 | 0 | 0 | 0 | $-\frac{e^{-c_5}\,(-1+e^{c_5})}{c_5}$ | 0 | 0 |
| t[7] | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{-1+e^{c_7}}{c_7}$ | 0 |
| t[9] | 0 | $-\frac{e^{-c_9}\,(-1+e^{c_9})}{c_9}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| t[11] | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{-1+e^{c_{11}}}{c_{11}}$ |
| t[13] | $-\frac{e^{-c_{13}}\,(-1+e^{c_{13}})}{c_{13}}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| t[15] | 0 | 0 | 0 | 0 | $\frac{-1+e^{c_{15}}}{c_{15}}$ | 0 | 0 | 0 |

```
{Alexander[K = Knot[8, 17]][X], Draw[GD[K]]}
```

$$\left\{11-\frac{1}{X^3}+\frac{4}{X^2}-\frac{8}{X}-8\,X+4\,X^2-X^3,\right.$$




$\left.\right\}$

```
β = Times @@ GD[K] /. {Ar[x_, y_, +1] :> R[x, y], Ar[x_, y_, -1] :> Rinv[x, y]};
Table[
  {k, (β = β // gm[1, k, 1]) // βForm, Collect[Last[β] /. t[i_] :> c_i, _h, Simplify]},
  {k, 2, 2 Crossings[K]}
 ] // ColumnForm
```

$\left\{2,\right.$

| 1 | h[1] | h[4] | h[6] | h[8] | h[10] | h[12] | h[14] | h[ |
|---|---|---|---|---|---|---|---|---|
| t[1] | 0 | 0 | $\frac{e^{-c_{13}}\,(-1+e^{c_1})}{c_1}$ | 0 | 0 | 0 | 0 | |
| t[3] | 0 | 0 | 0 | $-\frac{e^{-c_3}\,(-1+e^{c_3})}{c_3}$ | 0 | 0 | 0 | |
| t[5] | 0 | 0 | 0 | 0 | 0 | $-\frac{e^{-c_5}\,(-1+e^{c_5})}{c_5}$ | 0 | |
| t[7] | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{-1+e^{c_7}}{c_7}$ | |
| t[9] | 0 | $-\frac{e^{-c_9}\,(-1+e^{c_9})}{c_9}$ | 0 | 0 | 0 | 0 | 0 | |
| t[11] | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{-1}{c}$ |
| t[13] | $-\frac{e^{-c_{13}}\,(-1+e^{c_{13}})}{c_{13}}$ | 0 | $\frac{e^{-c_{13}}\,(-1+e^{c_1})\,(-1+e^{c_{13}})}{c_{13}}$ | 0 | 0 | 0 | 0 | |
| t[15] | 0 | 0 | 0 | 0 | $\frac{-1+e^{c_{15}}}{c_{15}}$ | 0 | 0 | |

$$\left\{3,\ \begin{array}{c|cccccccc} 1 & h[1] & h[4] & h[6] & h[8] & h[10] & h[12] & h[14] & h[ \\ t[1] & 0 & 0 & \frac{e^{-c_{13}}\,(-1+e^{c_1})}{c_1} & -\frac{e^{-c_1}\,(-1+e^{c_1})}{c_1} & 0 & 0 & 0 & \\ t[5] & 0 & 0 & 0 & 0 & 0 & -\frac{e^{-c_5}\,(-1+e^{c_5})}{c_5} & 0 & \\ t[7] & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+e^{c_7}}{c_7} & \\ t[9] & 0 & -\frac{e^{-c_9}\,(-1+e^{c_9})}{c_9} & 0 & 0 & 0 & 0 & 0 & \\ t[11] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{c} \\ t[13] & -\frac{e^{-c_{13}}\,(-1+e^{c_{13}})}{c_{13}} & 0 & \frac{e^{-c_{13}}\,(-1+e^{c_1})\,(-1+e^{c_{13}})}{c_{13}} & 0 & 0 & 0 & 0 & \\ t[15] & 0 & 0 & 0 & 0 & \frac{-1+e^{c_{15}}}{c_{15}} & 0 & 0 & \end{array}\right.$$

$$\left\{4,\ \begin{array}{c|cccccccc} 1 & h[1] & h[6] & h[8] & h[10] & h[12] & h[14] & h[16 \\ t[1] & 0 & \frac{e^{-c_9-c_{13}}\,(-1+e^{c_1})}{c_1} & -\frac{e^{-c_1-c_9}\,(-1+e^{c_1})}{c_1} & 0 & 0 & 0 & 0 \\ t[5] & 0 & 0 & 0 & 0 & -\frac{e^{-c_5}\,(-1+e^{c_5})}{c_5} & 0 & 0 \\ t[7] & 0 & 0 & 0 & 0 & 0 & \frac{-1+e^{c_7}}{c_7} & 0 \\ t[9] & -\frac{e^{-c_9-c_{13}}\,(-1+e^{c_9})}{c_9} & \frac{e^{-c_9-c_{13}}\,(-1+e^{c_1})\,(-1+e^{c_9})}{c_9} & -\frac{e^{-c_1-c_9}\,(-1+e^{c_1})\,(-1+e^{c_9})}{c_9} & 0 & 0 & 0 & 0 \\ t[11] & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+e^{c} \\ t[13] & -\frac{e^{-c_{13}}\,(-1+e^{c_{13}})}{c_{13}} & \frac{e^{-c_{13}}\,(-1+e^{c_1})\,(-1+e^{c_{13}})}{c_{13}} & 0 & 0 & 0 & 0 & 0 \\ t[15] & 0 & 0 & 0 & \frac{-1+e^{c_{15}}}{c_{15}} & 0 & 0 & 0 \end{array}\right.$$

$$\left\{5,\ \begin{array}{c|cccccccc} 1 & h[1] & h[6] & h[8] & h[10] & h[12] & h[14] & h[16 \\ t[1] & 0 & \frac{e^{-c_9-c_{13}}\,(-1+e^{c_1})}{c_1} & -\frac{e^{-c_1-c_9}\,(-1+e^{c_1})}{c_1} & 0 & -\frac{e^{-c_1}\,(-1+e^{c_1})}{c_1} & 0 & 0 \\ t[7] & 0 & 0 & 0 & 0 & 0 & \frac{-1+e^{c_7}}{c_7} & 0 \\ t[9] & -\frac{e^{-c_9-c_{13}}\,(-1+e^{c_9})}{c_9} & \frac{e^{-c_9-c_{13}}\,(-1+e^{c_1})\,(-1+e^{c_9})}{c_9} & -\frac{e^{-c_1-c_9}\,(-1+e^{c_1})\,(-1+e^{c_9})}{c_9} & 0 & 0 & 0 & 0 \\ t[11] & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+e^{c} \\ t[13] & -\frac{e^{-c_{13}}\,(-1+e^{c_{13}})}{c_{13}} & \frac{e^{-c_{13}}\,(-1+e^{c_1})\,(-1+e^{c_{13}})}{c_{13}} & 0 & 0 & 0 & 0 & 0 \\ t[15] & 0 & 0 & 0 & \frac{-1+e^{c_{15}}}{c_{15}} & 0 & 0 & 0 \end{array}\right.$$

$$\left\{6,\ \begin{array}{c|cccccc} e^{-c_9-c_{13}}\,(-1+e^{c_1}+e^{c_9+c_{13}}) & h[1] & h[8] & h[10] & h[12] & h[ \\ t[1] & \frac{e^{c_1-c_9-c_{13}}\,(-1+e^{c_1})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_1} & -\frac{e^{c_{13}}\,(-1+e^{c_1})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_1} & 0 & -\frac{e^{c_9+c_{13}}\,(-1+e^{c_1})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_1} & \\ t[7] & 0 & 0 & 0 & 0 & \frac{-1} \\ t[9] & -\frac{-1+e^{c_9}}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_9} & -\frac{e^{-c_1+c_{13}}\,(-1+e^{c_1})\,(-1+e^{c_9})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_9} & 0 & \frac{e^{-c_1}\,(-1+e^{c_1})^2\,(-1+e^{c_9})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_9} & \\ t[11] & 0 & 0 & 0 & 0 & \\ t[13] & -\frac{e^{c_9}\,(-1+e^{c_{13}})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_{13}} & \frac{e^{-c_1}\,(-1+e^{c_1})^2\,(-1+e^{c_{13}})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_{13}} & 0 & \frac{e^{-c_1+c_9}\,(-1+e^{c_1})^2\,(-1+e^{c_{13}})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_{13}} & \\ t[15] & 0 & 0 & \frac{-1+e^{c_{15}}}{c_{15}} & 0 & \end{array}\right.$$

$$\left\{7,\ \begin{array}{c|cccccc} e^{-c_9-c_{13}}\,(-1+e^{c_1}+e^{c_9+c_{13}}) & h[1] & h[8] & h[10] & h[12] & h[ \\ t[1] & \frac{e^{c_1-c_9-c_{13}}\,(-1+e^{c_1})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_1} & -\frac{e^{c_{13}}\,(-1+e^{c_1})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_1} & 0 & -\frac{e^{c_9+c_{13}}\,(-1+e^{c_1})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_1} & \frac{-1} \\ t[9] & -\frac{-1+e^{c_9}}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_9} & -\frac{e^{-c_1+c_{13}}\,(-1+e^{c_1})\,(-1+e^{c_9})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_9} & 0 & \frac{e^{-c_1}\,(-1+e^{c_1})^2\,(-1+e^{c_9})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_9} & \\ t[11] & 0 & 0 & 0 & 0 & \\ t[13] & -\frac{e^{c_9}\,(-1+e^{c_{13}})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_{13}} & \frac{e^{-c_1}\,(-1+e^{c_1})^2\,(-1+e^{c_{13}})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_{13}} & 0 & \frac{e^{-c_1+c_9}\,(-1+e^{c_1})^2\,(-1+e^{c_{13}})}{(-1+e^{c_1}+e^{c_9+c_{13}})\,c_{13}} & \\ t[15] & 0 & 0 & \frac{-1+e^{c_{15}}}{c_{15}} & 0 & \end{array}\right.$$

$$\left\{8,\ \begin{array}{c|ccc} e^{-c_9-c_{13}}\ (-1+e^{c_1}+e^{c_{13}}-e^{c_1+c_{13}}+e^{c_9+c_{13}}) & h[1] & h[10] & h[12] \\ t[1] & \frac{e^{-c_9-c_{13}}\ (-1+e^{c_1})\ (-1+e^{c_{13}})}{(1-e^{c_1}-e^{c_{13}}+e^{c_1+c_{13}}-e^{c_9+c_{13}})\ c_1} & 0 & \frac{e^{-c_1+c_9+c_{13}}\ (-1+e^{c_1})}{(1-e^{c_1}-e^{c_{13}}+e^{c_1+c_{13}}-e^{c_9+c_{13}})\ c_1} \\ t[9] & \frac{-1+e^{c_9}}{(1-e^{c_1}-e^{c_{13}}+e^{c_1+c_{13}}-e^{c_9+c_{13}})\ c_9} & 0 & \frac{e^{-c_1}\ (-1+e^{c_1})^2\ (-1+e^{c_9})\ (-1+e^{c_{13}})}{(1-e^{c_1}-e^{c_{13}}+e^{c_1+c_{13}}-e^{c_9+c_{13}})\ c_9} \\ t[11] & 0 & 0 & 0 \\ t[13] & -\frac{(-1+e^{c_1}-e^{c_9})\ (-1+e^{c_{13}})}{(1-e^{c_1}-e^{c_{13}}+e^{c_1+c_{13}}-e^{c_9+c_{13}})\ c_{13}} & 0 & -\frac{e^{-c_1+c_9}\ (-1+e^{c_1})^2\ (-1+e^{c_{13}})}{(1-e^{c_1}-e^{c_{13}}+e^{c_1+c_{13}}-e^{c_9+c_{13}})\ c_{13}} \\ t[15] & 0 & \frac{-1+e^{c_{15}}}{c_{15}} & 0 \end{array}\right\}$$

$$\left\{9,\ \begin{array}{c|ccc} e^{-c_1-c_{13}}\ (-1+e^{c_1}+e^{c_{13}}) & h[1] & h[10] & h[12] \\ t[1] & -\frac{e^{-c_1-c_{13}}\ (-1+e^{c_1})\ (-1+e^{c_{13}}+e^{c_1+c_{13}})}{(-1+e^{c_1}+e^{c_{13}})\ c_1} & 0 & -\frac{e^{-c_1}\ (-1+e^{c_1})\ \left(-1+2\ e^{c_1}-e^{2\,c_1}+e^{c_{13}}-e^{c_1+c_{13}}+e^{2\,c_1+c_{13}}\right)}{(-1+e^{c_1}+e^{c_{13}})\ c_1} \\ t[11] & 0 & 0 & 0 \\ t[13] & -\frac{-1+e^{c_{13}}}{(-1+e^{c_1}+e^{c_{13}})\ c_{13}} & 0 & \frac{(-1+e^{c_1})^2\ (-1+e^{c_{13}})}{(-1+e^{c_1}+e^{c_{13}})\ c_{13}} \\ t[15] & 0 & \frac{-1+e^{c_{15}}}{c_{15}} & 0 \end{array}\right\}$$

$$\left\{10,\ \begin{array}{c|cc} e^{-c_1-c_{13}}\ (-1+e^{c_1}+e^{c_{13}}) & h[1] & h[12] \\ t[1] & -\frac{e^{-c_1-c_{13}+c_{15}}\ (-1+e^{c_1})\ (-1+e^{c_{13}}+e^{c_1+c_{13}})}{(-1+e^{c_1}+e^{c_{13}})\ c_1} & -\frac{e^{-c_1+c_{15}}\ (-1+e^{c_1})\ \left(-1+2\ e^{c_1}-e^{2\,c_1}+e^{c_{13}}-e^{c_1+c_{13}}+e^{2\,c_1+c_{13}}\right)}{(-1+e^{c_1}+e^{c_{13}})\ c_1} \\ t[11] & 0 & 0 \\ t[13] & -\frac{-1+e^{c_{13}}}{(-1+e^{c_1}+e^{c_{13}})\ c_{13}} & \frac{(-1+e^{c_1})^2\ (-1+e^{c_{13}})}{(-1+e^{c_1}+e^{c_{13}})\ c_{13}} \\ t[15] & \frac{e^{c_1}\ (-1+e^{c_{15}})}{(-1+e^{c_1}+e^{c_{13}})\ c_{15}} & \frac{e^{-c_1}\ (-1+e^{c_1})\ \left(-1+2\ e^{c_1}-e^{2\,c_1}+e^{c_{13}}-e^{c_1+c_{13}}+e^{2\,c_1+c_{13}}\right)\ (-1+e^{c}}{(-1+e^{c_1}+e^{c_{13}})\ c_{15}} \end{array}\right\}$$

$$\left\{11,\ \begin{array}{c|cc} e^{-c_1-c_{13}}\ (-1+e^{c_1}+e^{c_{13}}) & h[1] & h[12] \\ t[1] & -\frac{e^{-c_1-c_{13}+c_{15}}\ (-1+e^{c_1})\ (-1+e^{c_{13}}+e^{c_1+c_{13}})}{(-1+e^{c_1}+e^{c_{13}})\ c_1} & -\frac{e^{-c_1+c_{15}}\ (-1+e^{c_1})\ \left(-1+2\ e^{c_1}-e^{2\,c_1}+e^{c_{13}}-e^{c_1+c_{13}}+e^{2\,c_1+c_{13}}\right)}{(-1+e^{c_1}+e^{c_{13}})\ c_1} \\ t[13] & -\frac{-1+e^{c_{13}}}{(-1+e^{c_1}+e^{c_{13}})\ c_{13}} & \frac{(-1+e^{c_1})^2\ (-1+e^{c_{13}})}{(-1+e^{c_1}+e^{c_{13}})\ c_{13}} \\ t[15] & \frac{e^{c_1}\ (-1+e^{c_{15}})}{(-1+e^{c_1}+e^{c_{13}})\ c_{15}} & \frac{e^{-c_1}\ (-1+e^{c_1})\ \left(-1+2\ e^{c_1}-e^{2\,c_1}+e^{c_{13}}-e^{c_1+c_{13}}+e^{2\,c_1+c_{13}}\right)\ (-1+e^{c}}{(-1+e^{c_1}+e^{c_{13}})\ c_{15}} \end{array}\right\}$$

$$\left\{12,\ \begin{array}{c} -e^{-2\,c_1-c_{13}}\ \left(e^{c_1}-e^{2\,c_1}-e^{c_1+c_{13}}+e^{c_{15}}-3\ e^{c_1+c_{15}}+3\ e^{2\,c_1+c_{15}}-e^{3\,c_1+c_{15}}-e^{c_{13}+c_{15}}+2\ e^{c_1+c_{13}+c_{15}}-2\ e^{2\,c_1+c_{13}+} \\ t[1] \\ t[13] \\ t[15] \end{array}\right\}$$

$$\left\{13,\ \begin{array}{cc} -e^{-3\,c_1}\ \left(e^{c_1}-2\ e^{2\,c_1}+e^{c_{15}}-4\ e^{c_1+c_{15}}+5\ e^{2\,c_1+c_{15}}-3\ e^{3\,c_1+c_{15}}+e^{4\,c_1+c_{15}}\right) & \\ t[1] & -\frac{e^{-3\,c_1}\ (-1+e^{c_1})\ \left(-e^{4\,c_1}+e^{2\,c_{15}}+e^{c_1+c_{15}}-e^{}\right.}{\left(e^{c_1}-2\ e^{2}\right.} \\ t[15] & -\frac{}{\left(e^{c_1}-2\ e^{2}\right.} \end{array}\right\}$$

$$\left\{14,\ \begin{array}{c} -e^{-3\,c_1}\ \left(1-3\ e^{c_1}+4\ e^{2\,c_1}-4\ e^{3\,c_1}+e^{4\,c_1}-e^{c_{15}}+4\ e^{c_1+c_{15}}-7\ e^{2\,c_1+c_{15}}+7\ e^{3\,c_1+c_{15}}-4\ e^{4\,c_1+c_{15}}+e^{5\,c_1+c_{15}}\right) \\ t[1] \\ t[15] \end{array}\right\}$$

$$\left\{15,\ \begin{array}{cc} -e^{-3\,c_1}\ \left(1-4\ e^{c_1}+8\ e^{2\,c_1}-11\ e^{3\,c_1}+8\ e^{4\,c_1}-4\ e^{5\,c_1}+e^{6\,c_1}\right) & h[1] \quad h[16] \\ t[1] & -\frac{e^{-c_1}\ (-1+e^{c_1})}{c_1} \quad \frac{-1+e^{c_1}}{c_1} \end{array}\right\},\ (-1+e^{-c_1})\ h[1]$$

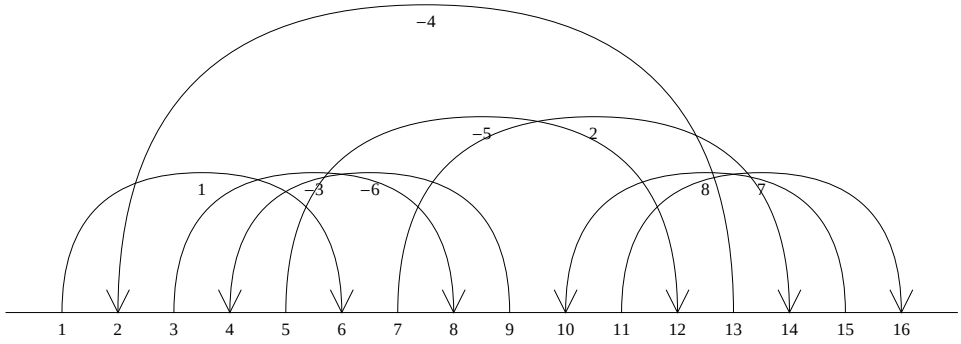
$$\left\{16,\ \left(\begin{array}{c} -e^{-2\,c_1}\ \left(1-4\ e^{c_1}+8\ e^{2\,c_1}-11\ e^{3\,c_1}+8\ e^{4\,c_1}-4\ e^{5\,c_1}+e^{6\,c_1}\right) \\ t[1] \end{array}\right),\ 0\right\}$$

*β*

$$B\left[-e^{-2c_1}\left(1-4e^{c_1}+8e^{2c_1}-11e^{3c_1}+8e^{4c_1}-4e^{5c_1}+e^{6c_1}\right),\ 0\right]$$

```
{Alexander[K = Knot[8, 17]][X], Draw[GD[K]]}
```

$$\left\{11-\frac{1}{X^3}+\frac{4}{X^2}-\frac{8}{X}-8X+4X^2-X^3,\right.$$




$\left.\right\}$

```
β = Times @@ GD[K] /. {Ar[x_, y_, +1] :> R[x, y],  Ar[x_, y_, -1] :> Rinv[x, y]};
indices = Range[2 Crossings[K]];
Table[
  j = RandomInteger[{1, k - 1}];
  β = β // gm[indices[[j]], indices[[j + 1]], indices[[j]]];
  indices = Delete[indices, j + 1];
  {indices, β // βForm; Collect[Last[β] /. t[i_] :> c_i, _h, Simplify]},
  {k, 2 Crossings[K], 2, -1}
 ] // ColumnForm
```

$\{\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16\}, (-1+e^{-c_{13}})\,h[2]+(-1+e^{-c_9})\,h[4]+(-1+e^{c_1})$
$\{\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 16\}, (-1+e^{-c_{13}})\,h[2]+(-1+e^{-c_9})\,h[4]+(-1+e^{c_1})\,h[6$
$\{\{1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 16\}, (-1+e^{-c_{13}})\,h[1]+(-1+e^{-c_9})\,h[4]+(-1+e^{c_1})\,h[6]+$
$\{\{1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16\}, (-1+e^{-c_{11}})\,h[1]+(-1+e^{-c_9})\,h[4]+(-1+e^{c_1})\,h[6]+(-1$
$\{\{1, 3, 4, 5, 6, 7, 8, 9, 10, 15, 16\}, (-1+e^{-c_{10}})\,h[1]+(-1+e^{-c_9})\,h[4]+(-1+e^{c_1})\,h[6]+(-1+e^{-}$
$\{\{1, 3, 4, 6, 7, 8, 9, 10, 15, 16\}, (-1+e^{-c_{10}})\,h[1]+(-1+e^{-c_9})\,h[4]+(-1+e^{c_1})\,h[6]+(-1+e^{-c_3})$
$\{\{1, 3, 4, 6, 7, 8, 9, 10, 15\}, (-1+e^{-c_{10}})\,h[1]+(-1+e^{-c_9})\,h[4]+(-1+e^{c_1})\,h[6]+(-1+e^{-c_3})\,h[8]$
$\{\{1, 3, 4, 7, 8, 9, 10, 15\}, (-1+e^{-c_{10}})\,h[1]+(-1+e^{c_1-c_9})\,h[4]+(-1+e^{-c_3})\,h[8]+(-1+e^{-c_4+c_7+c_{15}})$
$\{\{1, 4, 7, 8, 9, 10, 15\}, (-1+e^{-c_{10}})\,h[1]+(-1+e^{c_1-c_9})\,h[4]+(-1+e^{-c_1})\,h[8]+(-1+e^{-c_4+c_7+c_{15}})\,h[$
$\{\{1, 7, 8, 9, 10, 15\}, (-1+e^{c_1-c_9-c_{10}})\,h[1]+(-1+e^{-c_1})\,h[8]+(-1+e^{-c_1+c_7+c_{15}})\,h[10]+(-1+e^{c_{10}})\,h[$
$\left\{\{1, 7, 8, 9, 15\}, \left(-1+e^{c_1-2c_9}\right)h[1]+(-1+e^{-c_1})\,h[8]+(-1+e^{-c_1+c_7+c_{15}})\,h[9]+(-1+e^{c_9})\,h[15]\right\}$
$\left\{\{1, 8, 9, 15\}, \left(-1+e^{c_1-2c_9}\right)h[1]+(-1+e^{-c_1})\,h[8]+(-1+e^{c_{15}})\,h[9]+(-1+e^{c_9})\,h[15]\right\}$
$\left\{\{1, 9, 15\}, \left(-1+e^{-2c_9}\right)h[1]+(-1+e^{c_{15}})\,h[9]+(-1+e^{c_9})\,h[15]\right\}$
$\left\{\{1, 15\}, \left(-1+e^{-2c_1+c_{15}}\right)h[1]+(-1+e^{c_1})\,h[15]\right\}$
$\{\{1\}, 0\}$

*β* **// βForm**

$$\begin{pmatrix} -e^{-2c_1}\left(1-4e^{c_1}+8e^{2c_1}-11e^{3c_1}+8e^{4c_1}-4e^{5c_1}+e^{6c_1}\right) & h[1] \\ t[1] & 0 \end{pmatrix}$$